Last time: $f : W \to V$ linear function

$\mathrm{Ker}(f) \subseteq W$ are subspaces whose bases can be
$\mathrm{Im}(f) \subseteq V$ computed by Gaussian elimination

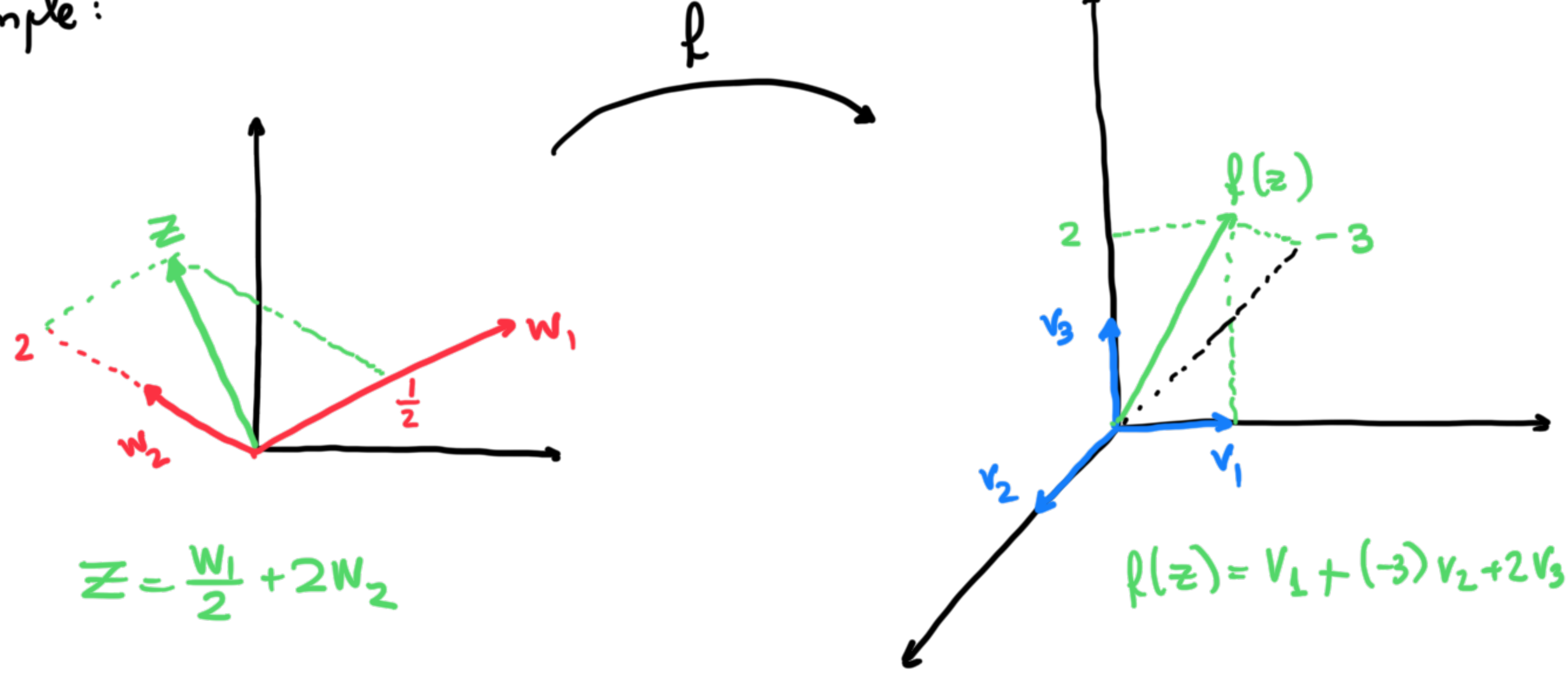
given bases $\underline{w} = (w_1, \dots, w_n)$ of $W$ and $\underline{v} = (v_1, \dots, v_m)$ of $V$,

the linear function $f$ is represented by a matrix $A \in \mathbb{R}^{m \times n}$

$$\Updownarrow$$

$$[f(z)]_{\underline{v}} = A\,[z]_{\underline{w}} \quad , \forall z \in W$$

Example:




$z = \frac{w_1}{2} + 2w_2$

$f(z) = v_1 + (-3)v_2 + 2v_3$

$A \in \mathbb{R}^{3\times 2}$ such that

Then $f$ is represented by a matrix $A \in \mathbb{R}$ such as

$$A \begin{pmatrix} \frac{1}{2} \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

$$[z]_{\underline{w}} \qquad\qquad [f(z)]_{\underline{v}}$$

Example: consider $V = W = \mathbb{P}_d = \{\text{polynomials of degree} \le d\}$

dim = d+1

and the bases $\underline{v} = \underline{w} = \{1, x, \ldots, x^d\}$

$$T : \mathbb{P}_d \to \mathbb{P}_d \quad , \quad T(\text{polynomial}) = \text{polynomial}'$$

Express $T$ by a matrix $A \in \mathbb{R}^{(d+1)\times(d+1)}$ w.r.t. monomial bases

$$\begin{array}{lcl}
w_1 = 1 & & T(w_1) = 0 = 0v_1 + 0v_2 + \ldots + 0v_{d+1} \\
w_2 = x & \xrightarrow{T} & T(w_2) = 1 = 1v_1 + 0v_2 + \ldots + 0v_{d+1} \\
w_3 = x^2 & & T(w_3) = 2x = 0v_1 + 2v_2 + \ldots + 0v_{d+1} \\
\vdots & & \vdots \\
w_{d+1} = x^d & & T(w_{d+1}) = dx^{d-1} = 0v_1 + \ldots + 0v_{d-1} + dv_d + 0v_{d+1}
\end{array}
\qquad
\begin{array}{l}
v_1 = 1 \\ v_2 = x \\ v_3 = x^2 \\ \vdots \\ v_{d+1} = x^d
\end{array}$$

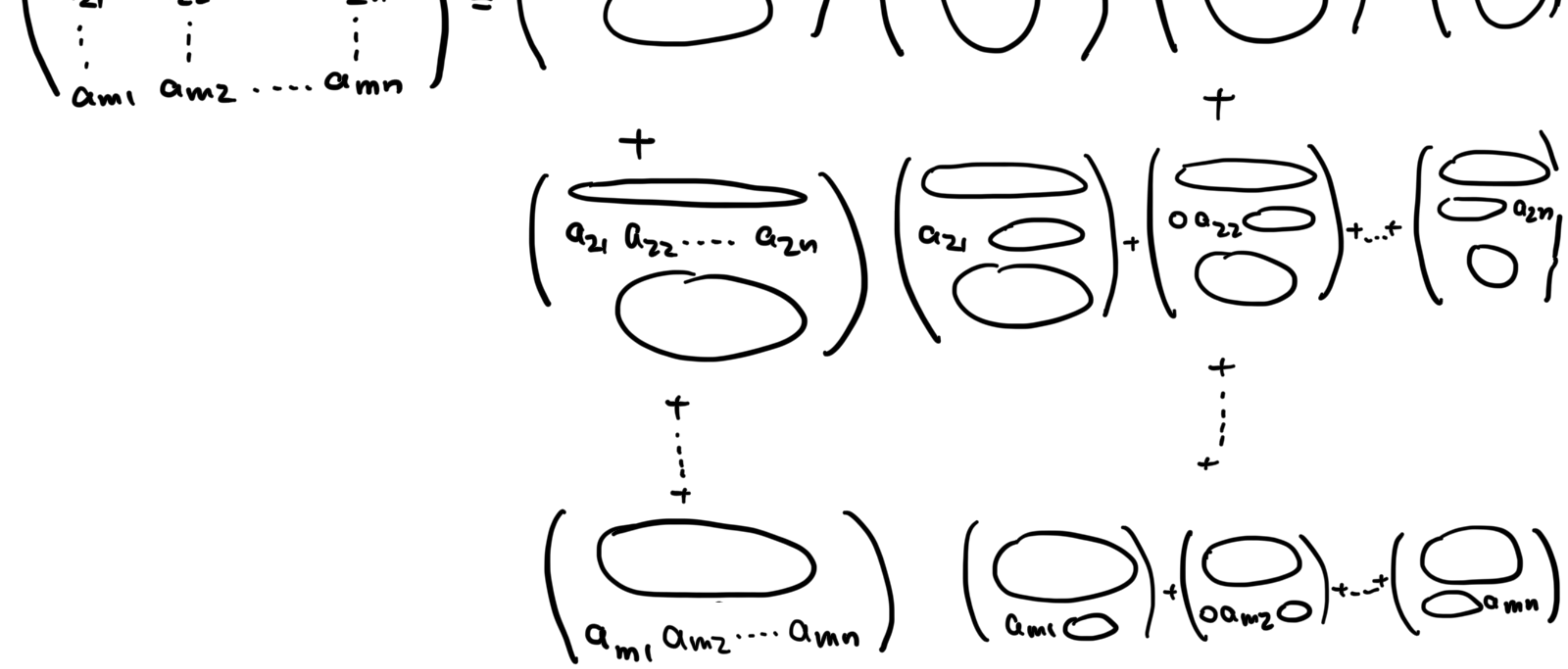
$$\Downarrow$$

$$A = \begin{pmatrix} \text{Exercise} \end{pmatrix} \in \mathbb{R}^{(d+1)\times(d+1)}$$

Example: let $V = \mathbb{R}^{m\times n} = \{m \times n \text{ matrices}\}$

dim $V = mn$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & & \end{pmatrix} = \begin{pmatrix} a_{11} & \\ & \end{pmatrix} + \begin{pmatrix} 0 & a_{12} \\ & \end{pmatrix} + \ldots + \begin{pmatrix} & a_{1n} \\ & \end{pmatrix}$$



$$= a_{11}E_{11} + \dots + a_{1n}E_{1n}$$
$$+ a_{21}E_{21} + \dots + a_{2n}E_{2n}$$
$$+$$
$$\vdots$$
$$+$$
$$+ a_{m1}E_{m1} + \dots + a_{mn}E_{mn}$$

$$E_{ij} = \begin{array}{c c} & j \\ i & \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & 1 & \cdots \\ 0 & \vdots & 0 \end{pmatrix} \end{array}$$

any matrix $A \in \mathbb{R}^{m \times n}$ can be written uniquely as a linear combination of $\{\underbrace{E_{ij} \in \mathbb{R}^{m\times n}}_{\text{a basis of } \mathbb{R}^{m\times n}}\}_{1 \le i \le m,\, 1 \le j \le n}$

e.g. $\begin{pmatrix} 2 & 3 \\ 9 & 5 \end{pmatrix} = 2E_{11} + 3E_{12} + 9E_{21} + 5E_{22}$

$T : \underset{\text{4-dim v.s}}{\mathbb{R}^{2\times2}} \longrightarrow \underset{\text{4-dim v.s.}}{\mathbb{R}^{2\times2}}$, $T(\underset{2\times2}{B}) = B^T$ a linear function

what matrix $A$ corresponds to the function $T$ w.r.t to the bases $\{E_{11}, E_{12}, E_{21}, E_{22}\}$ of $\mathbb{R}^{2\times2}$ ? Must have $A \in \mathbb{R}^{4\times4}$

$w_1 = E_{11}$ $\qquad$ $T(w_1) = E_{11} = 1v_1 + 0v_2 + 0v_3 + 0v_4$ $\qquad$ $v_1 = E_{11}$

$w_2 = E_{12}$ $\xrightarrow{T}$ $T(w_2) = E_{21} = 0v_1 + 0v_2 + 1v_3 + 0v_4$ $\qquad$ $v_2 = E_{12}$

$w_3 = E_{21}$ $\qquad$ $T(w_3) = E_{12} = 0v_1 + 1v_2 + 0v_3 + 0v_4$ $\qquad$ $v_3 = E_{21}$

$w_4 = E_{22}$ $\qquad$ $T(w_4) = E_{22} = 0v_1 + 0v_2 + 0v_3 + 1v_2$ $\qquad$ $v_4 = E_{22}$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

New topic:

**DEF 15.1:** An isomorphism is a bijective / invertible linear function $f: W \to V$

$$W \cong V$$ ("W is isomorphic to V")

- $V \cong V$
- if $V \cong W$ then $W \cong V$ (if $\exists f$ going one way, then $f^{-1}$ goes the other way)
- if $V \cong W$ and $W \cong Z$ then $V \cong Z$

**THM 15.2:** if $f: V \xrightarrow{\cong} W$ then

THM 15.2: if $f: V \to W$, then

- $v_1, \dots, v_n$ are linearly independent in $V$
  $\Updownarrow$
  $f(v_1), \dots, f(v_n)$ are linearly independent in $W$

- $v_1, \dots, v_n$ span / generate $V$
  $\Updownarrow$
  $f(v_1), \dots, f(v_n)$ span / generate $W$

- $v_1, \dots, v_n$ are a basis of $V$
  $\Updownarrow$
  $f(v_1), \dots, f(v_n)$ are a basis of $W$

  $\Rightarrow \dim(V) = \dim(W)$

based on the following fact:

$$v = c_1 v_1 + \dots + c_n v_n \iff f(v) = c_1 f(v_1) + \dots + c_n f(v_n)$$

THM 15.3: any vector space $V$ of dimension $n$ is isomorphic to $\mathbb{R}^n$

$\Downarrow$

up to isomorphism, any v.s. of dim 0 is $\mathbb{R}^0 = \{0\}$

any v.s. of dim 1 is $\mathbb{R}^1$

finite

any v.s. of dim 2 is $\mathbb{R}^2$

⋮

any v.s. of dim $\infty$ is "$\mathbb{R}^{\infty}$"

dim v.s.

Proof of Thm 15.3: $\dim(V) = n$

$\Downarrow$

$V$ has a basis $\underline{v} = \{v_1, \dots, v_n\}$

define $f : V \longrightarrow \mathbb{R}^n$, $\quad f(z) = [z]_{\underline{v}} \quad \forall z \in V$

Claim $\longrightarrow$ $f$ is linear, i.e. $f(c_1 z_1 + c_2 z_2) = [c_1 z_1 + c_2 z_2]_{\underline{v}} = c_1 [z_1]_{\underline{v}} + c_2 [z_2]_{\underline{v}} = c_1 f(z_1) + c_2 f(z_2)$

$\longrightarrow$ $f$ is injective, i.e. diff. $z \neq z'$ should have diff. coordinates

$\longrightarrow$ $f$ is surjective, i.e. $\forall \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$, $\exists\, z \in V$ s.t. $[z]_{\underline{v}} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$

$z = c_1 v_1 + \dots + c_n v_n$

$\Downarrow$

$f$ is an isomorphism

Ex: $V = \mathbb{R}^n \qquad \underline{v} = (v_1, \dots, v_n)$ an arbitrary basis of $\mathbb{R}^n$

$f : V \longrightarrow \mathbb{R}^n, \quad f(z) = [z]_{\underline{v}}, \quad \forall z \in V$

$V = \mathbb{R}^n$

$f(z) = Az$ for a matrix $A \in \mathbb{R}$

Claim: $A = P_{\underline{v}}^{-1} = P_{\underline{e} \leftarrow \underline{v}}^{-1} = P_{\underline{v} \leftarrow \underline{e}}$, because $\forall z \in \mathbb{R}^n$,

$$P_{\underline{v} \leftarrow \underline{e}}\, z = [z]_{\underline{v}}$$

$$\Updownarrow$$

$$z = P_{\underline{e} \leftarrow \underline{v}}\, [z]_{\underline{v}}$$

change of coordinate matrix from one basis to another

checks out for $z = v_i$

coordinates of $v_i = \begin{pmatrix} a_{1i} \\ \vdots \\ a_{ni} \end{pmatrix} =: P_{\underline{e} \leftarrow \underline{v}} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Big picture stuff: $W$ of dim $n$ $\rightsquigarrow$ basis $\underline{W} = \{w_1, \ldots, w_n\}$

$V$ of dim $m$ $\rightsquigarrow$ basis $\underline{V} = \{v_1, \ldots, v_m\}$

suppose you have a linear function $W \xrightarrow{f} V$

and you want to find the $m \times n$ matrix $A$ which represents it

$$\begin{tikzcd} \Phi: \mathbb{R}^n \arrow[r, bend left, "h^{-1}"] & W \arrow[l, "\cong"', "h"] \arrow[r, "f"] & V \arrow[r, "\cong", "g"'] & \mathbb{R}^m \end{tikzcd}$$

where $g: V \xrightarrow{\sim} \mathbb{R}^m$ $g(z) = [z]_{\underline{v}}$

where $g : V \xrightarrow{\sim} \mathbb{R}^m, \quad g(z) = [z]_{\underline{v}}$

$$h : W \xrightarrow{\sim} \mathbb{R}^n, \quad h(z) = [z]_{\underline{w}}$$

$$\Phi = g \circ f \circ h^{-1} : \mathbb{R}^n \to \mathbb{R}^m \quad \text{linear}$$

corresponds to $A \in \mathbb{R}^{m \times n}$; this is precisely the matrix which represents $f$

Ex:

$W = \mathbb{P}_2 \qquad \underline{w} = \{1, x, x^2\}$

$V = \mathbb{P}_3 \qquad \underline{v} = \{1, x, x^2, x^3\}$

$f : W \to V$, $f(\text{polynomial}) = \text{polynomial} \cdot (1+2x)$

$A \in \mathbb{R}^{4 \times 3}$; let's find $A$

$$\Phi : \mathbb{R}^3 \leftarrow \mathbb{P}_2 \xrightarrow{\text{mult by } (1+2x)} \mathbb{P}_3 \xrightarrow{\sim} \mathbb{R}^4$$

$e_1 \leftsquigarrow 1 \qquad\qquad 1 \rightsquigarrow e_1$

$e_2 \leftsquigarrow x \qquad\qquad x \rightsquigarrow e_2$

$e_3 \leftsquigarrow x^2 \qquad\qquad x^2 \rightsquigarrow e_3$

$\qquad\qquad\qquad\qquad x^3 \rightsquigarrow e_4$

$$e_1 \rightsquigarrow 1 \rightsquigarrow 1+2x \rightsquigarrow e_1 + 2e_2 = \Phi(e_1)$$

$$e_2 \rightsquigarrow x \rightsquigarrow x + 2x^2 \rightsquigarrow e_2 + 2e_3 = \Phi(e_2)$$

$$e_3 \rightsquigarrow x^2 \rightsquigarrow x^2 + 2x^3 \rightsquigarrow e_3 + 2e_4 = \Phi(e_3)$$

$$\text{so} \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$